IB132 - Fundations of Finance Notes*

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Contents

1 Prelude

Definition 1.1. Project A project is a set of cash flows. Usually projects entail an initial outflow such as investment, expense or cost, and are followed by a series of cash inflows like payoffs, revenues or returns.

Definition 1.2. Opportunity Cost Of Capital (OCC) The OCC is the rate at which our money can grow if we invest in other similar projects.

2 Present Value

2.1 Rate of Return

Definition 2.1. Rate of Return The rate of return from investing C_0 and getting C_1 after 1 period is defined as:

$$r = \frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1$$

2.2 Present Value (PV) Formula

Definition 2.2. Present Value Given cash flows $C_1, C_2, \ldots, C_T$ and interest rate r the present value is defined as:

$$PV = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

2.3 Net Present Value (NPV) Formula

Definition 2.3. Net Present Value Given cash flows $C_1, C_2, \ldots, C_T$ and interest rate r the net present value is defined as:

$$NPV = C_0 + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

Where C_0 is usually the initial cost of the project.

2.4 Capital Budgeting Rule

In a perfect market we should take all positive NPV projects.
Let $\phi(x)$ be a function such that $\phi(x) : \mathbb{R} \to \{0, 1\}$ then,

$$\phi(NPV) = \begin{cases} 1 \quad \text{(Accept Project)} & \text{if} \quad \text{NPV} \geq 0 \\ 0 \quad \text{(Reject Project)} & \text{if} \quad \text{NPV} < 0 \end{cases}$$

3 Perpetuities and Annuities

3.1 Simple Perpetuities

Definition 3.1. Perpetuity A perpetuity is a financial instrument that pays C pounds, per period, forever. Assuming constant r the PV of the perpetuity is:

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

3.2 Growing Perpetuities

Definition 3.2. Growing Perpetuity A growing perpetuity pays $C, C(1+g), C(1+g)^2, \ldots, C(1+g)^n, \ldots$. Assuming constant r and a constant growing rate of g the PV of the growing perpetuity is:

$$PV = \sum_{t=1}^{\infty} \frac{C \cdot (1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}$$

3.3 The Gordon Growth Model (GGM)

The Gordon Growth Model is an application of growing perpetuities and states that: Assume that a company will have profits of C pounds next year, which will grow at a rate g per year thereafter, forever. The cost of capital is r. Then, the firm value is given by:

$$\text{Business Value} = \frac{C}{r-g}$$

We can then use this formula to derive the price of a company stock today. Assume that we expect dividends from a firm to grow at a rate of g forever, and that the cost of capital is r. Further, dividends per-share will be D pounds next year then:

$$\text{Price Stock Today} = \frac{D}{r-g}$$

By rearranging equation the above equation we can also calculate a firm cost of capital via its dividend yield D/P, where D is the dividend per share next year and P is the price per share, in the following way:

$$r = \frac{D}{P} + g$$

3.4 Simple Annuities

Definition 3.3. Annuity An annuity is a financial instrument that pays C pounds, per period, for T periods. Assuming constant r the PV of the annuity is:

$$PV = \sum_{t=1}^{T} \frac{C}{(1+r)^t} = \frac{C}{r} \cdot \left(1 - \frac{1}{(1+r)^T}\right)$$

3.5 Growing Annuities

Definition 3.4. Growing Annuity A growing annuity pays $C, C(1+g), C(1+g)^2, \ldots, C(1+g)^T$. Assuming constant r and a constant growing rate of g the PV of the growing annuity is:

$$PV = \sum_{t=1}^{T} \frac{C \cdot (1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g} \cdot \left(1 - \frac{(1+g)^T}{(1+r)^T}\right)$$

4 Capital Budgeting

A firm has to choose between investing in different projects. The method that a firm uses to choose which projects to invest in is called a **decision rule** we have already discussed in section 1 on Present Values one possible decision rule. We shall now give different alternatives.

4.1 Internal Rate of Return (IRR)

Definition 4.1. IRR Is the value of the discount rate r which sets the NPV to 0 i.e. it is the zero-point solution of the equation $C_0 + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} = 0$.

4.1.1 Capital Budgeting Rule

Let $\phi(x)$ be a function such that $\phi(x) : \mathbb{R} \rightarrow \{0, 1\}$ then,

$$\phi(IRR) = \begin{cases} 1 & \text{(Accept Project)} \quad \text{if} \quad \text{IRR} \geq \text{hurdle rate} \\ 0 & \text{(Reject Project)} \quad \text{if} \quad \text{IRR} < \text{hurdle rate} \end{cases}$$

Where the *hurdle rate* is the cost of capital for the project as estimated by the manager.

4.1.2 Advantages

- Since calculating the cost of capital is a non-trivial computation, using the IRR can save time computation-wise since r does not figure in its formulation.
- The IRR allows for a margin of error when used as a decision rule. In fact, if the hurdle rate is different from the real cost of capital, we might still be making the right decision.

4.1.3 Disadvantages

- The IRR might not be unique.
- The IRR cannot handle different interest rates for different time horizons.
- The IRR is scale insensitive i.e. it ranks projects based on the rate of return return the size of the profit.

4.2 Profitability Index (PI)

Definition 4.2. PI It is the present value of all future cash flows divided by the absolute value of the cost:

$$PI = \frac{\sum_{t=1}^{T} \frac{C_t}{(1+r)^t}}{C_0} = \frac{PV}{Cost}$$

4.2.1 Capital Budgeting Rule

Let $\phi(x)$ be a function such that $\phi(x) : \mathbb{R} \rightarrow \{0, 1\}$ then,

$$\phi(PI) = \begin{cases} 1 & \text{(Accept Project)} \quad \text{if} \quad \text{PI} \geq 1 \\ 0 & \text{(Reject Project)} \quad \text{if} \quad \text{PI} < 1 \end{cases}$$

4.2.2 Disadvantages

- The PI is scale insensitive i.e. it ranks projects based on the rate of return return the size of the profit
- It does not keep the cost of capital separate.

4.3 Payback Rule

Definition 4.3. Payback It is the amount of time it takes to get back the money that has been invested.

4.3.1 Capital Budgeting Rule

Choose the project with the shortest payback time, or projects that payback sooner than some cut-off.

4.3.2 Disadvantages

- The payback rule ignores all cash flows beyond the cut-off point.
- It gives equal weight to all cash flows before the cut-off date. Hence, it ignores the time value of money.

5 Bonds

5.1 Assumptions

In this section we will assume the following:

- Markets are perfect.
- There is no uncertainty about future payoffs.

5.2 Background

Definition 5.1. Bond A bond is a security sold by government and corporations to raise money today in exchange for promised future payments.

Definition 5.2. Time To Maturity The time remaining until the maturity date is reached is called the time to maturity. Also called the **time** of the bond.

Definition 5.3. Yield To Maturity (YTM) The YTM is the discount rate - or internal rate of return (IRR) - that sets the NPV of the bond equal to zero.

5.2.1 Payments

Payments will occur until the **maturity date** of the bond. This vary from short term (one month) to long term (more than 10 years).

Bonds make two payments:

- Periodic interest payments called **coupons**.
- The principal or **face value (FV)**of the bond is repaid at maturity and is used to calculate the interest payments.

5.3 Zero Coupon Bonds

Definition 5.4. Zero Coupon Bond A zero coupon bond is a bond which does not make any payment apart from the its face value at the end of maturity.

5.3.1 Pricing

Definition 5.5. Price of a Zero Coupon Bond the price of a zero coupon bond is given by:

$$P = \frac{FV}{(1 + YTM_n)^n}$$

Where YTM_n is the annualized interest rate of return that would be received if the bond was held until period n which is the maturity of the bond.

5.4 Coupon Bonds

Definition 5.6. Coupon Bond A coupon bond is a bond which beside paying its face value, makes periodic interest payment - coupons.

5.4.1 Pricing

Definition 5.7. Price of a Coupon Bond The price of a coupon bond Is given by:

$$P = \underbrace{\frac{CPN}{YTM_n} \cdot \left(1 - \frac{1}{(1+YTM_n)^n}\right)}_{\text{Coupon payments (Annuity)}} + \underbrace{\frac{FV}{(1+YTM_n)^n}}_{\text{Face value payment}}$$

Where YTM_n is the annualized interest rate of return that would be received if the bond was held until period n which is the maturity of the bond.

5.5 Time and Bond Prices

Proposition 5.1. Suppose that we purchase an n-year zero coupon bond and we sell it after $m \leq n$ years then, the IRR of this project will always equal the Yield to Maturity (YTM) of the bond regardless of when we sell it, i.e. regardless of m.

Proof. If we bought this bond at time 0 and we sold it at time m, then the NPV of this project would be:

$$NPV = -\frac{FV}{(1+YTM)^n} + \frac{\frac{FV}{(1+YTM)^{n-m}}}{(1+IRR)^m}$$

Let us now compute the IRR of this project:

$$-\frac{FV}{(1+YTM)^n} + \frac{\frac{FV}{(1+YTM)^{n-m}}}{(1+IRR)^m} = 0 \implies$$

$$\frac{FV}{(1+YTM)^{n-m}} \cdot \left(-\frac{1}{(1+YTM)^m} + \frac{1}{(1+IRR)^m}\right) = 0 \implies$$

$$\frac{1}{(1+IRR)^m} = \frac{1}{(1+YTM)^m} \implies IRR = YTM$$

□

5.6 Interest Rates and Bond Prices

In general, **bond prices move inversely to interest rates**. This is due to the fact that a higher interest rate, increases the opportunity cost of capital for existing bonds; so, their cash flows are discounted at higher rates.

Further, it is possible to deduce the annual risk-free rate in an economy from a bond which is absolutely guaranteed to meet its promised payments - for example government bonds.

> **Example: Interest Rates and Bond Prices**
> Assume we have a 10-year government bond with $YTM = 10\%$ then, the risk-free rate is equal to the annualised YTM which in this case is$(1+0.1)^{1/10} - 1 \approx 0.95\%$.

5.7 Short vs. Long Term Bonds

We have seen that bond prices are affected by changes in the interest rate. However, not all bonds respond to this kind of changes in the same way. In fact, short horizon bonds have a smaller sensitivity to interest rates changes than longer term bonds.

> **Example: Short vs. Long Term Bonds**
> Suppose that I purchase a bond with FV = 100 which has a maturity of n years when the interest rate is at 8%
>
> - Suppose $n = 30$, then the price of the bond is $P_0 = 100/(1+0.08)^{30} \approx 9.94$
>   Now, if the interest rates increases by 1% -i.e. from 8% to 8.08% - immediately after we bought the bond, then the price of the bond decreases to $P_1 = 100/(1+0.0808)^{30} \approx 9.72$.

Further, if we were to sell this instrument now, we would walk away with a rate of return equal to $P_1/P_0 = 9.72/9.94 - 1 \approx -2.2\%$

- Suppose instead that $n = 1$ day, then the price of the bond is $P_0 = 100/(1+d) \approx 99.979$ where d is the daily interest rate which is equal to 0.0210%. Now, if the interest rates increases by 1% -i.e. from 8% to 8.08% - immediately after we bought the bond, then the price of the bond decreases to $P_1 = 100/(1+\hat{d}) \approx 99.978$ where $\hat{d} = 0.02129$.If we were to sell this instrument now, we would walk away with a rate of return equal to $P_1/P_0 = 99.978/98.979 - 1 \approx -0.001\%$

As it can be observed, the bond with maturity of 30 years yield a smaller return compared to the bond with maturity of 1 day in the case of an increase in the interest rate.

6 Uncertanty Defaulf and Risk

6.1 Assumptions

- Markets are perfect.
- Investors are risk neutral.

Howbeit, in this section, we will drop the assumption that there is no uncertainty in markets. Hence, we will learn how to make capital budgeting decisions when we face uncertainty about the outcomes of certain projects.

6.2 Expectation and Variance of a Radom Variable

Definition 6.1. Expectation Let X be a discrete random variable taking values $x_1, x_2, \dots x_n$ with probabilities $p_1, p_2 \dots p_n$ respectively. Then the expected value of this random variable is the finite sum

$$\mathbb{E}(X) = \sum_{i=1}^{n} x_i \cdot p_i$$

Definition 6.2. Variance Let X be a discrete random variable taking values $x_1, x_2, \dots x_n$ with probabilities $p_1, p_2 \dots p_n$ respectively and such that $\mathbb{E}(X) = \mu$. Then the variance of this random variable is the finite sum

$$Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 \cdot p_i$$

6.3 Risk attitude

We generally separate individuals into three group:

- **Risk averse** individuals do not accept an actuarially fair gamble.
- **Risk seeking** individuals accept an actuarially fair gamble.
- **Risk neutral** individuals are indifferent between accepting and rejecting an actuarially fair gamble.

As a consequence, if we want risk averse investors to accept an actuarially fair gamble we mush offer them a **risk premium**.

6.4 Default (or Credit) Risk

Most loans have credit risk because there is some non-zero probability that the issuer can default.Generally we can assume that the probability of default for developed economies is 0. However, other borrowers like developing countries, companies, banks, etc. have a non-zero probability of default.

Example: Short vs. Long Term Bonds

Suppose that we have a government issued bond B_G and a firm issued bond B_F with the following characteristics:

- B_G : $P = 200$ and promises an annual interest rate of 5%, i.e., we will get 210 in one year.
- B_F : $P = 200$ and promises 210 in one year. However, this company has a probability of 1% of defaulting on this loan, in which case you only receive 50.

What are the expected payment and rates of return on these bonds?

	$\mathbb{E}$(Payment)	$\mathbb{E}$(Return)
B_G	$100\% \times 210 = 210$	$100\% \times (\frac{210}{200} - 1) = 5\%$
B_F	$99\% \times 210 + 1\% \times 50 = 208.4$	$99\% \times (\frac{210}{200} - 1) + 1\% \times (\frac{50}{210} - 1) = 4.2\%$

So how much would you be willing to pay for B_F that promises to pay 210?(Recall that you are risk neutral). To answer this question, we solve $5\% = 99\%(\frac{210}{P_F} - 1) + 1\%(\frac{50}{P_F} - 1) \implies P_F \approx 198.48$. Further, at this price, the promised return -i.e. the return in the good state of the world- would be $\frac{210}{198.48} \approx 5.81\%$. As it can be observed, the promised return is 0.81% higher than the risk-free rate in this economy. Hence, we can interpret this difference as the default premium that makes the investor indifferent between buying a financial instrument which is 100% safe and one which has some probability of default.

More generally, given a financial instrument which promises to pay A pounds with probability ρ and B pounds with probability $(1 - \rho)$, in a world with risk neutral investor, the PV of this instrument would be

$$PV = \frac{\mathbb{E}(CashFlows)}{1 + OCC} = \frac{\rho \cdot A + (1 - \rho) \cdot B}{1 + r}$$

for this instrument. Where r is the risk-free rate in the economy which in this case is the correct OCC.

6.5 Premium Decomposition

In the example above, we have seen how in a market with risk neutral investors,

$$\text{Promised interest rate} \geq \mathbb{E}(\text{interest rate})$$

And we saw that if we want risk neutral investors to buy a financial instrument with some default risk embedded in it we would have to adjust the price of said instrument, and therefore its promised return, so that its

$$\text{Promised interest rate} = \text{Time premium} + \text{Default Premium}$$

Further,

$$\mathbb{E}(\text{interest rate}) = \text{Time Premium}$$

7 Uncertanty Bonds and Equity

7.1 Assumptions

- The world is risk neutral i.e., there are no risk premiums. This implies that all assets have the same cost of capital, and therefore yield the same return reflecting the time value of money.
- The market is perfect.

7.2 Financing Projects: Debt and Equity

Companies finance their projects using a mix of two instruments **Debt** and **Equity**.

- Debt (bond) holders: put up a certain amount of money today in exchange for a promised amount in the future.
  - Promises a pre-determined return.
  - Promised returns protected through covenants.
- Equity owners: receive whatever profits remain after bond holders are paid off.
  - Limited-liability security.
  - Residual claim on cash flow after creditors paid.
  - Shareholders receive return in the form of dividends and capital gains.

Definition 7.1. Levered Equity Equity issued by a firm with outstanding debt is called levered.

7.3 The State-contingent Payoff Table

		X		
State	Prob.	Project	Bond	Equity
Good	ρ	$C^{Proj.}_{Good}$	C^{Bond}_{Good}	C^{Equity}_{Good}
Bad	$1-\rho$	$C^{Proj.}_{Bad}$	C^{Bond}_{Bad}	C^{Equity}_{Bad}
$\mathbb{E}$(Value)		$V^X = \rho \cdot C^X_{Good} + (1-\rho) \cdot C^X_{Bad}$		
PV		$PV^{Proj.} = \frac{V^{Proj.}}{1+OCC}$	PV^{Bond}	$PV^{Proj.} - PV^{Bond}$
Return Good		$r^X_{Good} = \frac{C^X_{Good}}{PV^X}$		
Return Bad		$r^X_{Bad} = \frac{C^X_{Bad}}{PV^X}$		
$\mathbb{E}$(Return)		$r^X = \frac{V^X}{PV^X} - 1$		
Weight			$w_{Bond} = 1 - w_{Equity}$	$w_{Equity} = \frac{PV^{Equity}}{PV^{Proj.}}$

Where PV^{Bond} and PV^{Equity} is how much money we intend to finance the project by debt and by equity respectively - in this case, $PV^{Bond} + PV^{Equity} = PV^{Project}$ i.e. there is no other way to finance a project.

Note: if $w_{Bond} = 0$, then we say that the project is 100% financed by unlevered equity.

7.4 Comparison Between Debt Levels

Example: Statistics of Different Debt Levels

Consider a project which pays 200 or 110 with 80% and 20% probability respectively. Then, suppose we can choose between three different capital structures:

1. Low Debt: Suppose that we raise 71 from a bond issue with a promised rate of 2.86%.

2. Mid Debt: Suppose that we raise 131 from a bond issue for a promised rate of 16.9%.
3. High Debt: Suppose that we raise 142 from a bond issue for a promised rate of 24%.

How do they compare?

Debt level		Project	Debt	Equity
Low	$\mathbb{E}(r)$	20%	2.86%	35.07%
	SD	23.7%	0%	34.72%
Mid	$\mathbb{E}(r)$	20%	10.3%	81.3%
	SD	23.7%	13.2%	90.7%
High	$\mathbb{E}(r)$	20%	14.7%	98.0%
	SD	23.7%	18.7%	99.0%

Where, $SD^X = \sqrt{(r^X_{Good} - r^X)^2 \cdot \rho + (r^X_{Bad} - r^X)^2 \cdot (1-\rho)}$.

From the example above, we can draw the following generalisations:

- Increasing debt makes both debt and equity riskier. As a consequence, their respective cost of capital increases
- The overall riskiness of the project and cost of capital do not change.

8 Risk and Reward

In this section, we will shed light on how risk averse investors choose between different projects with uncertain payoffs. In doing so, we will engineer a better estimate for the opportunity cost of capital with which we will discount the payoffs of our projects.

8.1 Assumptions

- The market is perfect.

Nevertheless, we drop the assumption that investors are risk neutral and we allow for investors to be risk averse.

8.2 Risk Aversion

As we have previously seen, risk adverse investors are those who will reject a fair gamble. As a consequence, they have to be compensated with higher expected returns in order to accept riskier projects.This introduces a risk premium in expected returns of assets, in addition to the time and default premium.

Hence, in a market with risk averse investors:

$$\text{Promised interest rate} = \text{Time Premium} + \text{Default Premium} + \text{Risk Premium}$$

Further,

$$\mathbb{E}(\text{interest rate}) = \text{Time Premium} + \text{Risk Premium}$$

It is paramount to understand the risk/return profile of our projects to determine their appropriate cost of capital. A key insight is that investors are already holding several projects (i.e., stocks of other companies), i.e a portfolio. Hence, we can estimate the correct cost of capital for our projects by seeing how projects impact the portfolios that our investors are holding.

8.3 Covariance and Correlation

Definition 8.1. Covariance Let X and Y be two discrete random variables taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ and $y_1, y_2, \ldots, y_n$ with probabilities $q_1, q_2, \ldots, q_n$ respectively. Then the covariance of X and Y called $Cov(X, Y)$ is given by

$$Cov(X,Y) = \sum_{i=1}^{n} (x_i - \mu_x) \cdot (y_i - \mu_y) \cdot p_i$$

Where $\mu_x = \mathbb{E}(X)$ and $\mu_y = \mathbb{E}(Y)$.

Definition 8.2. Correlation Let X and Y be two discrete random variables taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ and $y_1, y_2, \ldots, y_n$ with probabilities $p_1, p_2, \ldots, p_n$ respectively. Then the correlation between X and Y called $Corr(X, Y)$ is given by

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \in [-1, 1]$$

- A correlation value close to 1 indicates that the two random variable more together in the same direction.
- A correlation value close to -1 indicates that the two random variable more together in the opposite directions.
- A correlation value close to 0 indicates that the two random variable do not move together.

8.4 Constructing a portfolio

Suppose now that we build a portfolio P consisting of two assets X and Y by investing $w_X\%$ of our wealth in X and $w_Y\%$ of our wealth in Y. Then, the expected rate of return of this portfolio would be given by

$$\mathbb{E}(r_P) = w_X \cdot \mathbb{E}(r_X) + w_Y \cdot \mathbb{E}(r_Y)$$

And the variance of its expected rate of return would be

$$\begin{aligned} Var(r_P) =& Var(r_X, r_Y) \\ =& w_X^2 \cdot Var(r_X) + w_Y^2 \cdot Var(r_Y) + 2 \cdot w_X \cdot w_Y \cdot Cov(r_X, r_Y) \end{aligned}$$

Where r_i indicates the rate of return of i.

From this, we can observe the benefit of diversification. In fact, if $Cov(X, Y) = 0$ then the variance of the rate of return on the portfolio will be lower than if $Cov(X, Y) > 0$ making the portfolio less volatile.

8.5 Diversification Benefits

Diversification benefits come from the fact that we are mixing together assets that are have less than perfect co-movement. More specifically, diversification eliminates **idiosyncratic risk** i.e. Market-wide shocks that affect all companies.

Definition 8.3. Idiosyncratic Risk Idiosyncratic risk is risk that is specific to an asset.It has little or no correlation with market risk, and can therefore be substantially mitigated from a portfolio through diversification.

Definition 8.4. Systematic Risk Systematic risk the is vulnerability to events which affect aggregate outcomes such as broad market. This kind or risk cannot be diversified away.

8.6 The Market Portfolio

Definition 8.5. Market Portfolio The market portfolio is a portfolio which provides the highest return per unit of risk taken on.

We assume that investors behave rationally and therefore recognize the benefits of diversification. As a consequence, they will all hold the market portfolio, i.e., those that maximize return for a given level of risk (albeit to different degrees).

8.7 Betas and Risk

Definition 8.6. Beta Beta is the association between a project, P, and the market portfolio, M, and it is given by

$$\beta_{P,M} = \frac{Cov(r_M, r_P)}{Var(r_M)}$$

- Projects with higher beta are riskier. Further, projects with higher betas are positively correlated to market movements, i.e. they have high returns when the market has high returns and have low returns when the market has low returns. Thus, they bring little or no diversification benefits which, in turn, makes them less appealing to investors. As a consequence, their prices are lower.
- Projects with negative beta are less risky. In a addition, projects negative betas are negatively correlated to market movements, i.e. they have high returns when the market has low returns and have low returns when the market has high returns. Thus, they bring diversification benefits which, in turn, makes them very appealing to investors. As a consequence, their prices are higher.

9 The Capital Asset Pricing Model (CAPM)

9.1 Assumptions

- The market is perfect.

9.2 The CAPM

Definition 9.1. CAPM The Capital Asset Pricing Model (CAPM) gives the appropriate expected return for any project given the risk of the project. Suppose that project i has a beta equal to β_i, then the CAPM states that

$$\mathbb{E}(r_i) = r_F + \beta_i \cdot (\mathbb{E}(r_M) - r_F)$$

Where:

- r_i is the rate of return of i.
- r_F is the risk-free rate of return.
- r_M is the risk rate of return of the market portfolio.

Note: the difference between the expected rate of return and the risk-free rate of return $(\mathbb{E}(r_M) - r_F)$ is called the equity premium or market risk premium

Then, if the CAPM holds, then the economy has the following characteristics:

- Idiosyncratic risk does not affect prices and returns.
- Systematic risk which is captured by β is the only kind of risk which affects prices and expected returns.

9.3 Law of One Price

Definition 9.2. Law of One Price The Law of one price states that projects with identical risk (same β) will have the same opportunity cost of capital, which is given by the CAPM.

10 Complications in Capital Budgeting

10.1 Tangible Complications

10.1.1 Transaction Costs

Example of transactions costs

Banks The bank incurs a cost to process loan applications, so lending rates $>$ borrowing rates.

Real Estate Agent The real estate agent advertises the property, etc, so charges a commission.

Market-Maker In stock markets the market-maker earns the bid-ask spread as compensation for providing immediate liquidity.
We sell at the bid and buy at the ask, and $price_{ask} > price_{bid}$

Let us give an example. Suppose that the bid quote on a corporate bond is 210 and the ask is 215. We expect this bond to return its promised return of 15% per annum for sure. If we liquidate our position in one month, what would be the return on your investment?

To answer this question, we first have to calculate the monthly promised return, and then we have to discount the ask price of the next month.

$$r = \frac{price_{bid} \cdot (1+m)}{price_{ask}} - 1 = \frac{210 \times (1+0.01171)}{215} - 1 \approx -1.18\%$$

Where m is the monthly promised return $m = (1+0.15)^{\frac{1}{12}} - 1$

10.1.2 Taxes

Taxes are applied only when earning positive returns, whereas transaction costs apply always.

Definition 10.1. Average Tax Rate

$$\text{Average Tax Rate} = \frac{\text{Total tax paid}}{\text{Total taxable income}}$$

Definition 10.2. Marginal Tax Rate The Marginal tax rate is the tax payable on the last pound of income.

We can calculate after-tax project return as:

$$r_{post-tax} = (1-\tau) \cdot r_{pre-tax}$$

Where τ is the marginal tax-rate

Example: Taxes and Net Present Value
A real-estate project costs 1M and produces 60K per year in ordinary taxable income for 10 years. After 10 years we can sell this project for 800K, which will be taxable at a rate of 20%. If your tax bracket is 33%, and taxable bonds offer a rate of 8% per annum, and equivalent tax-exempt ISAS offer 6% per annum, what is the NPV of the project?

$$NPV = \underbrace{-1M}_{\text{Cost}} + \underbrace{\frac{60K \times (1-0.33)}{0.06} \times \left(1 - \frac{1}{(1+0.06)^{10}}\right)}_{\text{PV Annuity}} + \underbrace{\frac{800K \times (1-0.20)}{(1+0.06)^{10}}}_{\text{PV Liquidation}}$$

10.1.3 Inflation

Let us denote the nominal rate of return as ϵ, the real rate as r and the rate of inflation as π. Then

$$(1+r) \cdot (1+\pi) = (1+\epsilon) \implies r = \frac{1+\epsilon}{1+\pi} - 1$$

10.2 Intangible Complications

The analysis so far proposed, relies on stock markets being efficient. This implies that investors are rational and therefore they:

- Properly considered all relevant information about the future cash flows
- Discount the future cash flows using a discount factor that reflects the systematic risk of the stock.
- Set prices accordingly.

10.2.1 Market Efficiency

Definition 10.3. Statistical CAPM

$$r_{i,t} - r_{F,t} = \underbrace{\alpha}_{\text{Abnormal return}} + \underbrace{\beta_{i,t-1} \cdot (r_{M,t} - r_{F,t})}_{\text{CAPM expected return}} + \underbrace{\epsilon_{i,t}}_{\text{Regression residuals}}$$

Where α represents abnormal returns, i.e., the average return that is unrelated to systematic risk.
Note: $\alpha = 0 \implies$ the market is perfect.

10.2.2 Types of Market Efficiency

- **Weak form**: Stock returns reflect historical information, such as past stock prices.
- **Semi-strong form**: Stock returns reflect historical and public information, such as:

Positively	Negatively
Earnings surprises	Company Price/Earnings ratios
Analyst earnings forecasts	Company size
Past 1 year returns	Past 3 year returns

- **Strong form**: Stock returns reflect historical, public and private information, which is information known only to insiders, like company executives.

11 Capital Structure in Perfect Markets

11.1 Assumptions

- The market is perfect.

11.2 Financing Projects

In section 7.2 we have seen how firms can finance their project through a mix of debt and equity. One natural question that arises is: what is the optimal mix? Further, does a specific capital structure affects the firm value?

In this section, we will introduce the idea that **in perfect markets, a firm value is equal to its PV regardless of how its projects are financed.**

11.3 Weighted Average Cost Of Capital ($WACC$)

Definition 11.1. $WACC$

$$WACC = w_{debt} \cdot \mathbb{E}(r_{Debt}) + w_{Equity} \cdot \mathbb{E}(r_{Equity}) = \mathbb{E}(r_{Firm})$$

11.4 Modigliani-Miller Irrelevance Theorem

Theorem 11.1. Modigliani-Miller (M-M) Irrelevance Theorem Consider two firms which are identical except for their financial structures. The first (Firm U) is unlevered: that is, it is financed by equity only. The other (Firm L) is levered: it is financed partly by equity, and partly by debt. The Modigliani Miller theorem states that the value of the two firms is the same.

A consequence of the Modigliani-Miller indifference theorem is that the firms weighted average cost of capital ($WACC$) does not depend on debt-to-firm value ratio, it only reflects the average riskiness of the cash flows to both shareholders and bond holders.

Example: WACC and the CAPM

Suppose that Firm X has equity with market value of 20M and debt with market value of 10M. The risk free rate is 8% and the expected return on the market portfolio is 18%, per year. The β of Firm X equity is 0.9 (debt has no risk). What is Firm X' COC assuming that we are in a perfect market?

By the CAMP, we know that the expected return on the equity is

$$\mathbb{E}(r) = r_F + \beta \cdot (\mathbb{E}(r_M) - r_F) = 8\% + 0.9 \times (18\% - 8\%) = 17\%$$

Thus,

$$WACC = \frac{10M}{30M} \times (8\%) + \frac{20M}{30M} \times (17\%) = 14\%$$

Further, by the M-M Irrelevance Theorem, the cost of capital of an identical firm with a different capital structure would be the same.

12 Capital Structure in Imperfect Markets

In the previous section, we have seen how the M-M Irrelevance Theorem provides an elegant theory which connects a firm' $WACC$ and its capital structure. Yet, in the real world, market are not perfect so, the M-M does not hold.

12.1 Capital Structure in an Imperfect World

In an imperfect world,

$$\text{Firm Value} = \text{Project Value} + \text{Financing Value.}$$

12.1.1 Taxes

Tax liabilities favour debt over equity financing.

Example: Tax liabilities: Debt vs. Equity Financing

Assume that we are running a simple firm characterised by the following parameters:

Investment Cost in Year 0	200
Before-tax Gross Payoff in Year 1	280
Before tax Profit in Year 1	80
Corporate Income Tax Rate (τ)	30%
Appropriate (after-tax) cost of capital from 0 to 1	12%

Now, consider two scenarios: Equity-Financing: 100% equity financed.

Taxable Profits	80
Tax Liability Next Year	$\tau \times 80 = 30\% \times 80 = 24$
Cash to Owners Next Year	$80 - 24 = 56$
Cash Flow Next Year	$280 - 24 = 256$
Cash to Equity Owners Next Year	$256 - 200 = 56$

Debt-Financing: 200 debt financing at 11%.

Interest Payments	$200 \times 11\% = 22$
Taxable Profits	$80 - 22 = 58$
Tax Liability Next Year	$\tau \times 58 = 30\% \times 58 = 17.40$
Cash Flow Next Year	$280 - 17.40 = 262.60$
Cash to Owners Next Year	$\underbrace{22}_{\text{Creditors}} + \underbrace{(58 - 17.40)}_{\text{Shareholders}} = 62.60$

Definition 12.1. Adjusted Present Value (APV) Suppose that the corporate income tax rate is given by τ then,

$$APV = \underbrace{\frac{\mathbb{E}(C^{All-Equity})}{1 + \mathbb{E}(r_{Firm})}}_{\text{PV of all-equity finaced firm}} + \underbrace{\frac{\tau \cdot \mathbb{E}(\overbrace{\text{Debt} \cdot r_{Debt}}^{\text{Interest Payment}})}{1 + \mathbb{E}(r_{Firm})}}_{\text{Expected Tax Savings}}$$

Definition 12.2. Adjusted $WACC$

$$\begin{aligned} WACC_{Adj.} =& \mathbb{E}(r_{Firm}) - \tau \cdot \mathbb{E}(r_{Debt}) \cdot w_{Debt} \\ =& w_{Debt} \cdot \mathbb{E}(r_{Debt}) \cdot (1 - \tau) + w_{Equity} \cdot \mathbb{E}(r_{Equity}) \end{aligned}$$

Thus,

$$PV = \frac{\mathbb{E}(C^{All-Equity})}{1 + WACC_{Adj.}}$$

Note: since $\tau < 1 \implies WACC_{Adj} < WACC \implies$ that tax adjusted $WACC$ discounts the firm's cash flows at a lower rate.

Example: Adjusted Present Value and $WACC_{Adj}$

A firm can invest in a project that costs 1000, and will generate a payoff of 1600 in one year. The appropriate after tax cost of capital for the firm is 15%. The firms tax rate is 40%.

Present value of the firm if it is entirely financed with equity:
Taxable Profits Next Year: $1600 - 1000 = 600$.
Taxes Paid Next Year: $\tau \times (600) = 0.4 \times (600) = 240$.
Value of the Firm Next Year: $\mathbb{E}(C^{All-Equity}) = 1600 - 240 = 1360$
Present Value fo the Firm:$PV = \frac{1600-240}{1.15} = 1182.6$.

Now suppose that the firm finances the project with 800 of debt at a cost of debt of 10%.Then the firm' Adjusted Present Value is:

$$\begin{aligned} APV =& \frac{\mathbb{E}(C^{All-Equity})}{1.15} + \frac{\tau \times (800 \times 10\%)}{1.15} \\ =& \frac{1360}{1.15} + \frac{40\% \times (800 \times 10\%)}{1.15} = 1210.40 \end{aligned}$$

Let us also compute it via the adjusted $WACC$:

$$PV = \frac{\mathbb{E}(C^{All-Equity})}{1 + WACC_{Adj.}} = \frac{\mathbb{E}(C^{All-Equity})}{1 + \mathbb{E}(r_{Firm}) - \tau \cdot \mathbb{E}(r_{Debt}) \cdot w_{Debt}}$$
$$= \frac{1360}{1 + 15\% - 40\% \times 10\% \times \frac{800}{1210.42}} = 1210.40$$

We indeed get the same result.

12.1.2 Discipline

Discipline problems favour debt over equity.

12.1.3 Information Asymmetry

Information asymmetry problems favour debt over equity.

12.1.4 Distress

Distress costs favour equity financing.

12.1.5 Misvaluation

Overpriced firms will prefer equity financing.
Underpriced ones debt financing.

12.2 Optimal Capital structure

The **pecking order theory** of financing suggests that managers prefer to finance projects in the following order:

1. with internal funds (i.e., retained earnings).
2. then with debt.
3. then with equity.

13 Equity Payout

Definition 13.1. Free Cash Flows Free Cash Flow is Cash flows over which managers have discretionary spending power.

Uses of free-cash flows

Free Cash Flow	Retain	Re-Invest
		Increase Cash Reserves
	Payout to Shareholders	Repurchase Shares
		Pay Dividends

13.1 Payout methods

13.1.1 Dividend payments

Important concepts:

- They can happen: Quarterly, semi-annually, or annually.

- Declaration date: board of directors authorizes the payment of a dividend.

- Cum-dividend: last date on which the shareholder has the right to receive the dividend.
- The next day, is called the ex-dividend date.

13.1.2 Repurchases

The firm uses cash to buy shares from its shareholders.
Important concepts:

- Auction Based Repurchases:
  Dutch Auction: The firm lists different prices at which it is prepared to buy shares, and shareholders in turn indicate how many shares they are willing to sell at each price. The firm then pays the lowest price at which it can buy back its desired number of shares
- Open-Market Repurchases:
  Tender Offer: A public announcement of an offer to buy back a specified amount of outstanding securities at a pre-specified price (typically set at a 10%-20% premium to the current market price) over a pre-specified period of time (usually about 20 days)

13.2 Payout policy in Perfect Markets

Theorem 13.1. Modigliani-Miller Dividend Policy Irrelevance In perfect capital markets, holding fixed the investment policy of a firm, the firms choice of dividend policy is irrelevant and does not affect the firm value.

13.3 Payout policy in Imperfect Markets

13.3.1 Taxes on Dividends and Capital Gains

In an imperfect market, shareholders must pay taxes on the dividends they receive, and capital gains taxes when they sell their shares at profit. However, dividends are typically taxed at a higher rate than capital gains. As a consequence, firms are reluctant to raise funds to pay dividends.

13.3.2 Optimal Dividend Policy with Taxes

When the tax rate on dividends is greater than the tax rate on capital gains, shareholders will pay lower taxes if a firm uses share repurchases rather than dividends, so rational investors should prefer to receive money via repurchases as opposite to dividends. Therefore, all else being equal, firms that payout with share repurchases should have a higher value compared to firms that payout with dividends, because investment in these companies is worth more.

Thus, if investors are assumed to be rational, the optimal dividend policy, when the dividend tax rate exceeds the capital gain tax rate, is to pay no dividends at all.

13.4 Dividend Puzzle

In imperfect markets, payout policy can affect firm value exactly in the same way that capital structure does. We have seen how dividends should not be chosen over share repurchases when taxes are taken into account. Yet, we still observe dividends being paid. Why is that?

The reason lies in the following arguments which can be made for dividends:

- Dividends are more regular so can discipline managers.

- Increasing dividends may signal to investors that the firm will be profitable in the future.
- Due to non-rational reasons investors may prefer dividends.

14 Option Contracts

14.1 Background

Definition 14.1. Financial Option A contract that provides the owner with the right - but not the obligation - to purchase or sell an asset (such as a stock) at a fixed price at a future date is called an option. Further, we say that an option has been exercised when the owner enforces the agreement, and buys or sells the option at the agreed-upon price.

Definition 14.2. Call Option A contract that provides the owner with the right - but not the obligation - to buy an asset at some future date at some specified price is called a call option.

Definition 14.3. Put Option A contract that provides the owner with the right - but not the obligation - to sell an asset at some future date at some specified price is called a put option.

Definition 14.4. Option Writer The agent which sells a financial options is called the option writer.

Definition 14.5. Strike or Exercise Price The pre-specified price at which the option holder buys or sells the asset when the option is exercised is called the strike or exercise price.

Definition 14.6. Expiration Date The last date on which the option holder has the right to exercise their option is called the expiration date.

Definition 14.7. American Option Contracts which allow their holders to exercise the right to buy or sell an option on any date up to the expiration date are called American options.

Definition 14.8. European Option Contracts which allow their holders to exercise the right to buy or sell an option only on the the expiration date are called European options.

Definition 14.9. Open Interest The total number of contracts of a particular option that have been written is known as the open interest of that option.

Definition 14.10. At-the-Money Describes an option whose exercise price is equal to the current stock price.

Definition 14.11. In-the-Money Describes an option whose value if immediately exercised would be positive.

Definition 14.12. Out-of-the-money Describes an option whose value if immediately exercised would be negative.

We say that the option buyer has a long position in an option contract. Whereas the option seller has a short position in the contract.

14.2 Options and Profitability

Throughout the rest of this section, let S denote the stock price at expiration, K the strike price, and P the option price at expiration.

14.2.1 Profitability of Call Options

When the stock price is higher than the strike price, that is when $S > K$, exercise the call option - buy the stock at K - and then sell it of S on the open market. The quantity $S - K$ will determine our profit. On the other hand, if at the expiration date $S \leq K$ then the call option would be worth 0. Thus, the price of a call option is defined as

$$P = max\{S - K, 0\}$$

14.2.2 Profitability of Put Options

When the stock price is lower than the strike price, that is when $S < K$, buy the stock on the open market for S and then exercise the put option - that is sell the stock at K. The quantity $K - S$ is our profit. On the other hand, if at the expiration date $S \leq K$ then the put option would be worth 0. Thus, the price of a call option is defined as

$$P = max\{K - S, 0\}$$

14.3 Option Combinations

Options can be used to hedge risk, i.e., to reduce risk by holding contracts or securities whose payoffs are negatively correlated with some other risk exposure.

Definition 14.13. Protective Put A long position in a put option held on a stock you already own is called a protective put.

Definition 14.14. Protective Call A long position in a call option held on a bond you already own is called a protective call.

Protective puts and calls are kind of portfolio insurance.

14.4 Put-Call Parity

Because both protective calls and protective puts provide exactly the same payoff, the Law of One Price requires that they must have the same price. Thus,

$$S + P = PV(K) + C,$$

where S is the strike price of the option (the price you want to ensure that the stock will not drop below), C is the call price, P is the put price, and C is the call price. Further, rearranging the terms gives an expression for the price of a European call option for a non-dividend-paying stock:

$$C = S + P - PV(K).$$

This relationship between the stock price, the bond and the put and call options is known as put-call parity.

14.5 Factors Affecting Option Prices

The value of a call option:

- increases (decreases) as the strike price decreases (increases), all other things held constant.
- increases (decreases) as the stock price increases (decreases), all other things held constant.

The value of a put option:

- The value of a put option increases (decreases) as the strike price increases (decreases), all other things held constant.
- The value of a put option increases (decreases) as the stock price decreases (increases), all other things held constant.

Moreover,the value of an option generally increases with the volatility of the underlying stock.

Example: Options and Stock Volatility

Suppose that we have two European call options with the same strike price K but, written on two different stocks X and Y. Assume that the expected price of both stocks is the same, and it is equal to K. Further, let the volatility of stock X be lower then the volatility of stock Y, i.e. $SD(X) < SD(Y)$, and assume that stock X, having low volatility, will be worth K with probability 1. Now, since we have assumed that stock Y had a greater volatility, suppose that

Y's expected value is equal to $\rho(K + \epsilon) + (1 - \rho)(K - \epsilon) = K$ for $0 \leq \rho \leq 1$ and $\epsilon > 0$. Which option will be worth more today?

Since we have assumed that the strike price is K for both calls and that stock X will be worth K for sure, the option whose underlying asset is X will be worth $K - K = 0$. On the other hand, the option whose underlying asset is Y has a positive probability of being worth $(K + \epsilon) - K = \epsilon$, which by assumptions is greater than 0.

14.6 Option Valuation:Black-Scholes Model

Fisher Black and Myron Scholes in 1972 conceived the following equations to value option contracts.

Given a call option written on a non-dividend paying stock, its value is given by:

$$C = S \cdot N(d_1) - PV(K) \cdot N(d_2),$$

given a put option written on a non-dividend paying stock, its value is given by:

$$P = PV(K) \cdot (1 - N(d_2)) - S \cdot (1 - N(d_1)),$$

where S is the current stock price, K is the strike price, $PV(K)$ is the price of a zero-coupon bond that pays K on the date that the option expires, and $N(d)$ is the cumulative normal distribution which shows the probability that a normally distributed variable takes a value less than d. Further, we take

$$d_1 = \frac{ln(S/PV(K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where σ is the annual volatility (standard deviation), and T is the number of years left to expiration.

Example: The Black-Scholes Model in Practice

Suppose that the MediaSet SpA share price is currently 1.475. The annual standard deviation is 35%. A call option on MediaSet expires in three months, with a exercise price of 1.00. The risk-free rate is 4% per year. MediaSet is not expected to pay dividends during this period. Let us use the Black-Scholes formula to calculate the price of the call option.

We start off by calculating the monthly risk-free rate (since the option expires in months and not years): $r = (1 + 0.01)^{1/12} - 1 = 0.00327$. Further, since the option expires in three months, T, which is defined to be the number of years left eo expiration, is equal to 3/12. Then, we can calculate $PV(K) = K/(1 + r)^3 = 0.99$. Now

$$d_1 = \frac{ln(1.475/0.99)}{0.35\sqrt{3/12}} + \frac{0.35\sqrt{3/12}}{2}, \quad d_2 = d_1 - 0.35\sqrt{3/12}.$$

Which yields $d_1 = 2.36$ and $d_2 = 2.19$. Now, via a statistical table or a computer, we can calculate the values of $N(d_1)$ and $N(d_2)$ which are 0.9909 and 0.9857, respectively. Thus, putting all the bits together, we get that the value of the call option is

$$C = 1.475 \cdot 0.9909 - 0.99 \cdot 0.9857 = 0.48.$$